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Quantum algebraic description of the Moszkowski model

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Abstract. In this paper we investigate the behaviour of the Moszkowski model within the context of quantum algebras. The Moszkowski Hamiltonian is diagonalized exactly for different numbers of particles and for various values of the deformation parameter of the quantum algebra involved. We also include cranking in our system and observe its variation as a function of the deformation parameters.

1. Introduction

Quantum algebras [1], also known as quantum universal enveloping (QUE) algebras are generalizations of the usual Lie algebras, differing from them in the associativity condition. Instead of the usual Jacobi identity necessary to identify a Lie algebra, the quantum algebras are required to satisfy a Yang–Baxter equation, also known as a braid-Jacobi equation. QUE algebras are also called Hopf algebras. From the physical point of view, they can describe deformations of systems previously studied within the context of Lie algebras, i.e. they can describe perturbations from some underlying symmetry structure. Stretching effects are taken into account when one allows the algebra to deviate from the usual Lie algebra limit by means of a deformation parameter. It is worth pointing out that the connection of QUE algebras to q-groups is quite different from the usual link established between Lie algebras and Lie groups.

In order to study the many body problem microscopically, many exactly solvable models have been developed and utilized in investigating more realistic theories [2–3]. Applications of quantum algebras to some of these models which obey a Lie algebra structure [4] may help us to understand such features as symmetry breaking or phase transitions. In this paper we study the behaviour of a system described by the Moszkowski Hamiltonian [3] when deviations from the $\mathrm{su}(2) \times \mathrm{su}(2)$ algebra are introduced. The Moszkowski model allows one to study the transition from the vibrational (single-particle motion) to the rotational (collective motion) regime in the atomic nucleus. We are interested in learning how the physical properties (namely equilibrium energy, excitation energies, phase transitions) of particles described by

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this schematic model change when they are subject to *quantum deformations*. For instance, the energy gap between the first excited state and the ground state is one of the quantities which allows the verification of the existence of the phase transition in this model. One may wonder whether this phase transition is preserved when deviations from the underlying $su(2) \otimes su(2)$ algebra are imposed; this point will be discussed in a later section.

2. The Moszkowski model

2.1. *The original model*

The Moszkowski model is a two-level model, each level being N-fold degenerate with two different kinds of particles, i.e. N_a particles of type a occupying two levels and N_b particles of type b occupying two other levels. The model is the two-dimensional analogue of the Elliott model [5], which also includes a one-body spin-orbit term. The $su(2) \times su(2)$ Hamiltonian which describes the model reads

$$H = \epsilon(J_z(a) - J_z(b)) + V(J_x^2 + J_y^2) \tag{1}$$

where ϵ is the energy difference between both levels, V is the interaction

$$J_i = J_i(a) + J_i(b) \qquad J^2 = \sum_i J_i^2 \qquad i = x, y, z \tag{2}$$

and

$$J_x(\alpha) = \frac{1}{2}(J_+(\alpha) + J_-(\alpha)) \qquad J_y(\alpha) = -\frac{i}{2}(J_+(\alpha) - J_-(\alpha)) \qquad \alpha = a, b. \tag{3}$$

The quasi-spin operators J_z, J_+ and J_- are related to the quantum numbers which describe states of particles a and b. These quantum numbers are $\sigma = \pm 1/2$, having the value $1/2$ in the upper level and $-1/2$ in the lower level and p (appearing below) specifying the particular degenerate state within the level. In this way, J_z, J_+ and J_- are defined as

$$\begin{aligned} J_z(\alpha) &= \sum_p \tfrac{1}{2}(\alpha^\dagger_{p,+1/2}\alpha_{p,+1/2} - \alpha^\dagger_{p,-1/2}\alpha_{p,-1/2}) \\ J_+(\alpha) &= \sum_p \alpha^\dagger_{p,+1/2}\alpha_{p,-1/2} \\ J_-(\alpha) &= \sum_p \alpha^\dagger_{p,-1/2}\alpha_{p,+1/2} \end{aligned} \tag{4}$$

where $\alpha = a, b$, the creation operators $a^\dagger_{p,\pm 1/2}$ ($b^\dagger_{p,\pm 1/2}$) create a particle of type a (b) in the state p with $\sigma = \pm\frac{1}{2}$, and $a_{p,\pm 1/2}$ ($b_{p,\pm 1/2}$) are the corresponding annihilation operators. The operators J_+, J_- and J_z satisfy the following commutation relations

$$\begin{aligned} [J_+(\alpha), J_-(\beta)] &= 2J_z(\alpha)\delta_{\alpha\beta} \\ [J_z(\alpha), J_\pm(\beta)] &= \pm J_\pm(\alpha)\delta_{\alpha\beta} \end{aligned} \tag{5}$$

and can be used in rewriting the Hamiltonian given in equation (1) yielding

$$H = \epsilon(J_z(a) - J_z(b)) + \tfrac{1}{2}V(J_+(a)J_-(a) + J_-(a)J_+(a) + J_+(b)J_-(b) + J_-(b)J_+(b) + 2J_+(a)J_-(b) + 2J_+(b)J_-(a)). \tag{6}$$

Notice that the above Hamiltonian commutes with J_z, but does not commute with J^2. The basis of states on which the Moszkowski Hamiltonian can be diagonalized exactly is given by

$$|\psi_{ab}\rangle = |\tfrac{1}{2}N_a m_a\rangle|\tfrac{1}{2}N_b 2m_b\rangle \tag{7}$$

where

$$m_a = -\frac{N_a}{2}, -\frac{N_a}{2}+1, \ldots, \frac{N_a}{2} \qquad m_b = -\frac{N_b}{2}, -\frac{N_b}{2}+1, \ldots, \frac{N_b}{2}. \tag{8}$$

Some features of the Moszkowski model have already been extensively discussed [6–7], but our aim in this work is somewhat different. Here we discuss the behaviour of the model within the context of quantum algebras.

2.2. *The deformed model*

The Hamiltonian of the Moszkowski model can be written in the same form as the one shown in equation (6), but the quasi-spin operators J_+, J_- and J_z are now generators of the of the $\mathrm{su}_q(2)$ quantum algebra and, instead of obeying equations (5), they satisfy the following commutation relations

$$\begin{aligned} &[J_+(\alpha), J_-(\beta)] = [2J_z(\alpha)]\delta_{\alpha\beta} \\ &[J_z(\alpha), J_\pm(\beta)] = \pm J_\pm(\alpha)\delta_{\alpha\beta} \end{aligned} \tag{9}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{10}$$

and q is the *deformation* parameter of the algebra. When $q \to 1$, $[x] = x$. Within this formalism, the application of the raising, lowering and $J_z(\alpha)$ operators to a generic ket of the deformed basis $|JM\rangle$ gives [8]

$$\begin{aligned} &J_z(\alpha)|JM\rangle = M|JM\rangle \\ &J_+(\alpha)|JM\rangle = \sqrt{[J-M][J+M+1]}|JM+1\rangle \\ &J_-(\alpha)|JM\rangle = \sqrt{[J+M][J-M+1]}|JM-1\rangle. \end{aligned} \tag{11}$$

The Hamiltonian we call the $(\mathrm{su}(2) \times \mathrm{su}(2))_q$-Moszkowski Hamiltonian can be diagonalized exactly with the help of equations (11). Notice that, whether it is deformed or not, the operator $J_z(\alpha)$ always has M as its eigenvalue, this being the state on which it is acting. In the original Moszkowski model J_a in $|J_a M_a\rangle$ is associated with the total number of particles of type a ($J_a = N_a/2$) and J_z with the difference of occupation between levels $\sigma = 1/2$ and $\sigma = -1/2$ (see equation (4)) and the same is valid concerning b particles. This interpretation is maintained in the deformed case. The deformation parameter simulates a new kind of interaction, but does not affect the way particles a and b are taken into account. At this point, we should also say that the deformation parameter can be written as $q = 1 + \tau$, where τ can be either real or imaginary and it is responsible for simulating a residual interaction which can be either attractive or repulsive. The behaviour of this residual interaction may depend on whether τ is real or imaginary. In what follows, it is always considered to be real.

3. Discussion of the results

In figure 1(*a*) we show the difference between the first excited state and the ground state as a function of NV/ϵ, where $N = N_a + N_b$ and $N_a = N_b = 8$ for $q = 1.0$, 1.2 and 2.0. In figure 1(*b*) we have $N_a = N_b = 30$ and show curves for $q = 1.0$, 1.1, 1.5 and 2.0. In both figures one may observe that when the interaction V is turned off, $(E_1 - E_0)/\epsilon$ is always equal to 1.0, independent of the number of particles considered. This fact can be easily understood since, in this case, the Hamiltonian just depends on $J_z(a)$ and $J_z(b)$ where, as already discussed, the deformation parameter plays no rôle. Notice from this figure that the inclusion of small deformations anticipates the phase transition that would happen in the $N_a + N_b$ system for larger interactions, and for larger systems, i.e. with more particles, this phase transition is more pronounced for the same value of the deformation parameter. However, for larger deformations, there appears a critical q_c above which the phase transition is suppressed. At this point it is worth emphasizing that whenever q_c is reached, the physics of the deformed system changes qualitatively in relation to the original model and this critical deformation parameter is reached faster (smaller q_c) in systems with more particles. We believe this is due to the fact that when q_c is reached, the interaction makes all particles very strongly correlated and the single-particle motion completely disappears. Observation of figure 1 indicates that τ $(q = 1 + \tau)$ simulates an attractive residual interaction which gradually makes the motion of the system become more and more collective.

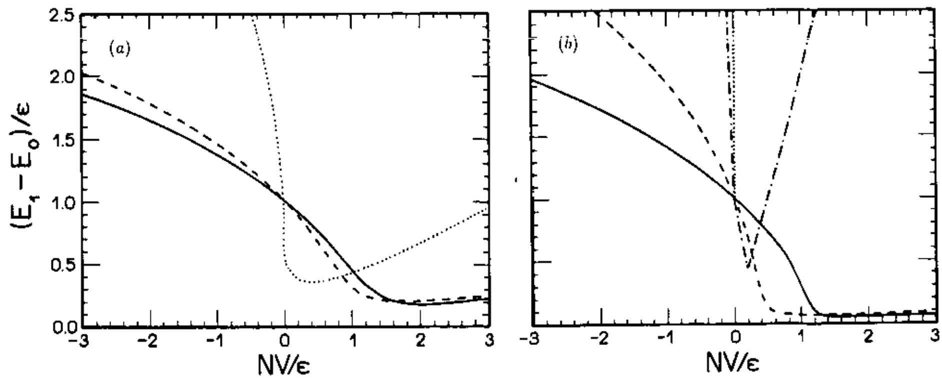

Figure 1. (*a*) Difference between the first excited state and the ground state as a function of NV/ϵ, where $N = N_a + N_b$ and $N_a = N_b = 8$ particles. The full curve shows the function for $q = 1.0$, the broken curve for $q = 1.2$ and the dotted curve for $q = 2.0$. (*b*) As (*a*), but $N_a = N_b = 30$ particles. The full curve shows the function for $q = 1.0$, the broken curve for $q = 1.1$, the chain curve for $q = 1.5$ and the dotted curve for $q = 2.0$.

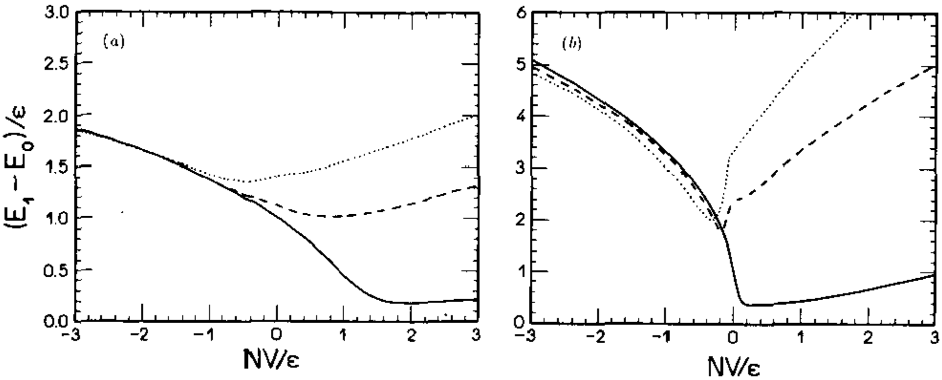

Figure 2. (*a*) Difference between the first excited state and the ground state as a function of NV/ϵ, where $N = N_a + N_b$ and $N_a = N_b = 8$ particles for $q = 1.0$. The full curve shows the function for $\omega = 0.0$, the broken curve for $\omega = 0.5$ and the dotted curve for $\omega = 1.0$. (*b*) As (*a*) but for $q = 2.0$.

Following the idea developed in [7], we introduce cranking as a way of imposing a symmetry breaking in the system considered. For this purpose we add the term $-\omega J_x$ to the Hamiltonian shown in equation (6). This choice was made in contrast to the one in [7] $(-wJ_z)$ because in our model, J_z is already the symmetry axis (i.e. $[H, J_z] = 0$). The *cranked* Hamiltonian is given by

$$H_{\text{crank}} = H - \frac{\omega}{2}(J_+(a) + J_-(a) + J_+(b) + J_-(b)). \tag{12}$$

The cranked Hamiltonian, for $N_a = N_b = 8$ particles, is diagonalized for different values of ω and the difference between the first excited state and the ground state as a function of NV/ϵ is shown in figure 2(*a*) for $q = 1.0$ and in figure 2(*b*) for $q = 2.0$. For $\omega = 0$, the solid curves in figures 2(*a*) and 2(*b*) are the same as the solid and dotted curves shown in figure 1(*a*). For the original Moszkowski system shown in figure 2(*a*), the introduction of cranking (i.e. when $\omega = 0.5$ or $\omega = 1.0$) completely suppresses the phase transition. Nevertheless, in the deformed system, the phase transition is never suppressed so long as the deformation parameter remains small for the number of particles under consideration, as can be seen in figure 2(*b*).

Another point investigated was the behaviour of the ground state as a function of the J_z projection for different values of q. The results we obtained are shown in figure 3 for $N_a = N_b = 30$ particles and $q = 1.0$, 1.1, 2.0 when the interaction is kept fixed at $V = +2$. The general trend of the curves seems to remain the same but, for larger q we obtain higher minima. The critical deformation parameter q_c, already mentioned, also plays its rôle here. When it is reached, both sides of the curve close together.

To summarize, we would like to say that, in this work, the Moszkowski model has been analysed in the context of the q-deformed $\mathrm{su}(2) \otimes \mathrm{su}(2)$ algebra in order to check the effects of the deformation parameter on the physical properties of systems which can be described by this toy model. We have focused our attention on the first two energy levels and their difference, which give new insights on the behaviour of the phase transition.

Some work on the same lines as this paper has already been done, namely the application of quantum algebras to the su(2) Lipkin model [9]. Generalizations from

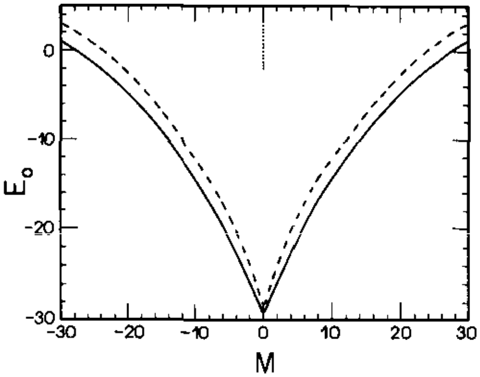

Figure 3. Ground state plotted against M, which is the projection of the J_z operator, for $q = 1.0$ (full curve), $q = 1.1$ (broken curve) and $q = 2.0$ (dotted curve) for $N_a = N_b = 30$ particles and $V = +2$.

the classical to the quantum su(N) algebra have also been studied [10] and they can be useful in helping to extend the above application to su(3), or even the su(N) Lipkin model. This work is already under investigation.

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